A CONVERSE TO MOORE'S THEOREM ON CELLULAR AUTOMATA

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ABSTRACT. We prove a converse to Moore's "Garden-of-Eden" theorem: a group G is amenable if and only if all cellular automata living on G that admit mutually erasable patterns also admit gardens of Eden.

It had already been conjectured in [11; 1, Conjecture 6.2] that amenability could be characterized by cellular automata. We prove the first part of that conjecture.

1. INTRODUCTION

Definition 1.1. Let G be a group. A finite *cellular automaton* on G is a map $\theta : Q^S \to Q$, where Q, the *state set*, is a finite set, and S is a finite subset of G.

Note that usually G is infinite; much of the theory holds trivially if G is finite. S could be taken to be a generating set of G, though this is not a necessity.

A cellular automaton should be thought of as a highly regular animal, composed of many cells labeled by G, each in a state $\in Q$. Each cell "sees" its neighbours as defined by S, and "evolves" according to its neighbours' states.

More formally: a *configuration* is a map $\phi : G \to Q$. The *evolution* of the automaton $\theta : Q^S \to Q$ is the self-map $\Theta : Q^G \to Q^G$ on configurations, defined by

$$\Theta(\phi)(x) = \theta(s \mapsto \phi(xs)).$$

Two properties of cellular automata received special attention. Let us call *patch* the restriction of a configuration to a finite subset $Y \subseteq G$. On the one hand, there can exist patches that never appear in the image of Θ. These are called *Garden of Eden* (GOE), the biblical metaphor expressing the notion of paradise lost forever.

On the other hand, Θ can be non-injective in a strong sense: there can exist patches $\phi'_1 \neq \phi'_2 \in Q^Y$ such that, however one extends ϕ'_1 to a configuration ϕ_1, if one extends ϕ'_2 similarly (i.e. in such a way that ϕ_1 and ϕ_2 have the same restriction to $G \setminus Y$) then $\Theta(\phi_1) = \Theta(\phi_2)$. These patches ϕ'_1, ϕ'_2 are called *Mutually Erasable Patterns* (MEP). Equivalently[1] there are two configurations ϕ_1, ϕ_2 which differ on a non-empty finite set, with $\Theta(\phi_1) = \Theta(\phi_2)$. The absence of MEP is sometimes called *pre-injectivity*.

Cellular automata were initially considered on $G = \mathbb{Z}^n$. Celebrated theorems by Moore and Myhill [55, 566, 6] prove that, in this context, a cellular automaton admits GOE if and only if it admits MEP. This result was generalized by Machì

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[1]In the non-trivial direction, let ϕ_1, ϕ_2 differ on a non-empty finite set F; set $Y = F(S \cup S^{-1})$ and let ϕ'_1, ϕ'_2 be the restriction of ϕ_1, ϕ_2 to Y respectively.

and Mignosi [33, 3] to G of subexponential growth, and by Ceccherini, Machì and Scarabotti [11, 1] to G amenable.

We prove that this last result is essentially optimal, and yields a characterization of amenable groups:

Theorem 1.2. *Let G be a group. Then the following are equivalent:*

(1) *the group G is amenable;*
(2) *all cellular automata on G that admit MEP also admit GOE.*

Schupp had already asked in [77; 7, Question 1] in which precise class of groups the Moore-Myhill theorem holds.

Ceccherini et al. write in [11, 1][2]:

Conjecture 1.3 ([11; 1, Conjecture 6.2])**.** *Let G be a non-amenable finitely generated group. Then for any finite and symmetric generating set S for G there exist cellular automata θ_1, θ_2* with that *S such that*

- *In θ_1 there are MEP but no GOE;*
- *In θ_2 there are GOE but no MEP.*

As a first step, we will prove Theorem 1.2, in which we allow ourselves to choose an appropriate subset S of G. Next, we extend a little the construction to answer the first part of Conjecture 1.3:

Theorem 1.4. *Let $G = \langle S \rangle$ be a finitely generated, non-amenable group. Then there exists a cellular automaton $\theta : Q^S \to Q$ that has MEP but no GOE.*

We conclude that the property of "satisfying Moore's theorem" is independent of the generating set, a fact which was not obvious *a priori*.

2. Proof of Theorem 1.2

The implication (1) $\Rightarrow$ (2) has been proven by Ceccherini et al.; see also [22; 2, §8] for a slicker proof. We prove the converse.

Let us therefore be given a non-amenable group G. Let us also, as a first step, be given a large enough finite subset S of G. Then there exists a "bounded propagation $2:1$ compressing vector field" on G: a map $f : G \to G$ such that $f(x)^{-1}x \in S$ and $\#f^{-1}(x) = 2$ for all $x \in G$.

We construct the following automaton θ. Its stateset is

$$Q = S \times \{0, 1\} \times S.$$

Order S in an arbitrary manner, and choose an arbitrary $q_0 \in Q$. Define $\theta : Q^S \to Q$ as follows:

(2.1)

$$\theta(\phi) = \begin{cases} (p, \alpha, q) & \text{for the minimal pair } s < t \text{ in } S \text{ with} \begin{cases} \phi(s) = (s, \alpha, p), \\ \phi(t) = (t, \beta, q), \end{cases} \\ q_0 & \text{if no such } s, t \text{ exist.} \end{cases}$$

[2] I changed slightly their wording to match this paper's

2.1. **Θ is surjective.** Namely, θ does not admit GOE. Let indeed ϕ be any configuration. We construct a configuration ψ with $\Theta(\psi) = \phi$.

Consider in turn all $x \in G$; write $\phi(x) = (p, \alpha, q)$, and $f^{-1}(x) = \{xs, xt\}$ for some $s, t \in S$ ordered as $s < t$. Set then

$$\psi(xs) = (s, \alpha, p), \qquad \psi(xt) = (t, 0, q). \tag{2.2}$$

Note that $\psi(z) = (f^{-1}(z)z, *, *)$ for all $z \in G$. Since $\#f^{-1}(z) = 2$ for all $z \in G$, it is clear that, for every $x \in G$, there are exactly two $s \in S$ such that $\psi(xs) = (s, *, *)$; call them s, t, ordered such that $\psi(xs) = (s, \alpha, p)$ and $\psi(xt) = (t, 0, q)$. Then $\Theta(\psi)(x) = (p, \alpha, q)$, so $\Theta(\psi) = \phi$.

2.2. **Θ is not pre-injective.** Namely, θ admits MEP. Let indeed $\phi : G \to Q$ be any configuration; then construct ψ following (2.2), and define ψ' as follows. Choose any $y \in G$, write $\phi(y) = (p, \alpha, q)$, and write $f^{-1}(y) = \{ys, yt\}$ for some $s, t \in S$, ordered as $s < t$. Define $\psi' : G \to Q$ by

$$\psi'(x) = \begin{cases} \psi(x) & \text{if } x \neq yt, \\ (t, 1, q) & \text{if } x = yt. \end{cases}$$

Then ψ and ψ' differ only at yt; and $\Theta(\psi) = \Theta(\psi')$ because the value of β is unused in (2.1). We conclude that θ has MEP.

3. Proof of Theorem 1.4

We begin by a new formulation of amenability for finitely generated groups:

Lemma 3.1. *Let G be a finitely generated group. The following are equivalent:*

(1) *the group G is not amenable;*
(2) *for every generating set S of G, there exist $m > n \in \mathbb{N}$ and a "$m : n$ compressing correspondence on G with propagation S"; i.e. a function $f : G \times G \to \mathbb{N}$ such that*

$$\forall y \in G : \quad \sum_{x \in G} f(x, y) = m, \tag{3.1}$$

$$\forall x \in G : \quad \sum_{y \in G} f(x, y) = n, \tag{3.2}$$

$$\forall x, y \in G : \quad f(x, y) \neq 0 \Rightarrow y \in xS. \tag{3.3}$$

Note that this definition generalizes the notion of "2 : 1 compressing vector field" introduced above.

Proof. For the forward direction, assuming that G is non-amenable, there exists a rational $m/n > 1$ such that every finite $F \subseteq G$ satisfies

$$\#(FS) \geq m/n\#F.$$

Construct the following bipartite oriented graph: its vertex set is $G \times \{1, \dots, m\} \sqcup G \times \{-1, \dots, -n\}$. There is an edge from (g, i) to $(gs, -j)$ for all $s \in S$ and all $i \in \{1, \dots, m\}, j \in \{1, \dots, m\}$. By hypothesis, this graph satisfies: every finite $F \subseteq G \times \{1, \dots, m\}$ has at least $\#F$ neighbours. Since $m > n$ and multiplication by a generator is a bijection, every finite $F \subseteq G \times \{-1, \dots, -n\}$ also has at least $\#F$ neighbours.

We now invoke the Hall-Rado theorem [44, 4]: if a bipartite graph is such that every subset of any of the parts has as many neighbours as its cardinality, then

there exists a “perfect matching” — a subset I of the edge set of the graph such that every vertex is contained in precisely one edge in I. Set then

$$f(x,y) = \#\{(i,j) \in \{1,\dots,m\}\times\{1,\dots,n\} : I \text{ contains the edge from } (x,i) \text{ to } (y,-j)\}.$$

For the backward direction: if G is amenable, then there exists an invariant measure on G, hence on bounded natural-valued functions on G. Let f be a bounded-propagation $m:n$ compressing correspondence; then

$$m = \sum_{x\in G}\int_{\{x\}\times G} f = \sum_{y\in G}\int_{G\times\{y\}} f = n,$$

contradicting $m > n$. □

Let now $G = \langle S\rangle$ be a non-amenable group, and apply Lemma 3.1 to $G = \langle S^{-1}\rangle$, yielding $m > n \in \mathbb{N}$ and a contracting $m:n$ correspondence f. Consider the following cellular automaton θ, with stateset

$$Q = (S\times\{0,1\}\times S^n)^n.$$

Choose $q_0 \in Q$, and give a total ordering to $S\times\{1,\dots,n\}$.

Consider $\phi \in Q^S$. To define $\theta(\phi)$, let $(s_1,k_1) < \dots < (s_m,k_m)$ be the lexicographically minimal sequence in $(S\times\{1,\dots,n\})^m$ such that

$$\phi(s_j)_{k_j} = (s_j,\alpha_j,t_{j,1},\dots,t_{j,n}) \in S\times\{0,1\}\times S^n \quad \text{for } j = 1,\dots,m.$$

If no such $s_1,k_1,\dots,s_m,k_m$ exist, set $\theta(\phi) = q_0$; otherwise, set

$$\theta(\phi) = ((t_{1,1},\alpha_1,t_{2,1},\dots,t_{n+1,1}),\dots,(t_{1,n},\alpha_n,t_{2,n},\dots,t_{n+1,n})) \in Q. \tag{3.4}$$

The same arguments as before apply. Given $\phi : G \to Q$, we construct $\psi : G\to Q$ such that $\Theta(\psi) = \phi$, as follows. We think of the coördinates $\psi(x)_k$ of $\psi(x)$ as n “slots”, initially all “free”. By definition, $\#f^{-1}(x) = m$ for all $x \in G$, while $\#f(x) = n$. Consider in turn all $x\in G$; write $f^{-1}(x) = \{xs_1,\dots,xs_m\}$, and let $k_1,\dots,k_m \in \{1,\dots,n\}$ be “free” slots in $\psi(xs_1),\dots,\psi(xs_m)$ respectively. By the definition of f, there always exist sufficiently many free slots.

Mark now these slots as “occupied”. Reorder $s_1,k_1,\dots,s_m,k_m$ in such a way that $(s_1,k_1,\dots,s_m,k_m)$ is minimal among its $m!$ permutations. Set then

$$\psi(xs_j)_{k_j} = (s_j,\alpha_j,t_{j,1},\dots,t_{j,n}) \quad \text{for } j = 1,\dots,m,$$

where $\alpha_{n+1},\dots,\alpha_m$ are taken to be arbitrary values (say 0 for definiteness) and

$$\phi(x) = ((t_{1,1},\alpha_1,t_{2,1},\dots,t_{n+1,1}),\dots,(t_{1,n},\alpha_1,t_{2,n},\dots,t_{n+1,n})).$$

Finally, define ψ arbitrarily on slots that are still “free”.

It is clear that $\Theta(\psi) = \phi$, so θ does not have GOE. On the other hand, θ has MEP as before, because the values of α_j in (3.4) are not used for $j \in \{n+1,\dots,m\}$.

4. Remarks

4.1. ***G*-sets.** A cellular automaton could more generally be defined on a right G-set X. There is a natural notion of amenability for G-sets, but it is not clear exactly to which extent Theorem 1.2 can be generalized to that setting.

4.2. **Myhill's Theorem.** It seems harder to produce counterexamples to Myhill's theorem (“GOE imply MEP”) for arbitrary non-amenable groups, although there exists an example on $C = C_2 * C_2 * C_2$, due to Muller[3]. Let us make our task even

[3]University of Illinois 1976 class notes

harder, and restrict ourselves to linear automata over finite rings (so we assume Q is a module over a finite ring and the map $\theta : Q^S \to Q$ is linear). The following approach seems promising.

Conjecture 4.1 (Folklore? I learnt it from V. Guba)**.** *Let G be a group. The following are equivalent:*

(1) *The group G is amenable;*

(2) *Let $\mathbb{K}$ be a field. Then $\mathbb{K}G$ admits right common multiples, i.e. for any $\alpha, \beta \in \mathbb{K}G$ there exist $\gamma, \delta \in \mathbb{K}G$ with $\alpha\gamma = \beta\delta$ and $(\gamma, \delta) \neq (0, 0)$.*

The implication (1) $\Rightarrow$ (2) is easy, and follows from Følner's criterion of amenability by linear algebra.

Assume now the "hard" direction of the conjecture. Given G non-amenable, we may then find a finite field $\mathbb{K}$, and $\alpha, \beta \in \mathbb{K}G$ that do not have a common right multiple.

Set $Q = \mathbb{K}^2$ with basis (e_1, e_2), let S contain the inverses of the supports of α and β, and define the cellular automaton $\theta : Q^S \to Q$ by

$$\theta(\phi) = \sum_{x \in G} \big(\alpha(x^{-1})\langle\phi(x)|e_1\rangle - \beta(x^{-1})\langle\phi(x)|e_2\rangle, 0\big).$$

Then θ has GOE, indeed any configuration not in $(\mathbb{K} \times 0)^G$ is a GOE. On the other hand, if θ had MEP, then by linearity we might as well assume $\Theta(\phi) = 0$ for some non-zero finitely-supported $\phi : G \to Q$. Write $\phi = (\gamma, \delta)$ in coördinates; then $\Theta(\phi) = 0$ gives $\alpha\gamma = \beta\delta$, showing that α, β actually did have a common right multiple.

Muller's example is in fact a special case of this construction, with

$$G = \langle x, y, z | x^2, y^2, z^2 \rangle,$$

$\mathbb{K} = \mathbb{F}_2$, and $\alpha = x$, $\beta = y + z$.

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